Flory approximant for the fractal dimension of the viscous-finger pattern: Transient and asymptotic behaviors

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We present a "Flory-type" mean-field theory to calculate the fractal dimension D of the radial viscous finger pattern formed by the less viscous fluid in a Hele-Shaw cell. We predict (a) the relation between the interfacial tension σ between the two fluids and the onset of the asymptotic behavior and (b) the variation of the effective fractal dimension with time in the transient regime for a fixed σ.

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Viscous-finger patterns are formed when a less viscous fluid drives a more viscous fluid in a Hele-Shaw cell, which consists of two closely placed mutually parallel plates [1]. Viscous fingering serves as a prototype for the study of the formation and growth of fingerlike interfacial patterns, also called curved fronts, observed in a wide variety of systems far from equilibrium [2]. In this work we shall consider only the radial viscous-finger patterns (RVFP); these are formed when the less viscous fluid is injected through a narrow hole at the center of the upper plate of the Hele-Shaw cell. The linear stability analysis for the circular interface [3] is similar to that for the planar interfaces [4]; the injection speed tends to destabilize the interface and competes against the stabilizing effects of the interfacial tension σ. In this paper we study the RVFP on length scales much longer than the finger width λ.

The fractal dimension D of the area A covered by the less viscous fluid in the asymptotic regime (i.e., $A \to \infty$) is a characteristic of the RVFP. If the two fluids in the Hele-Shaw cell are completely miscible [5] (i.e., if σ vanishes) the tips of the fingers continue to split, as the pattern evolves with time, leading to a pattern whose fractal dimension is identical to that of the diffusion-limited aggregates (DLA's), d_{DLA} [6]. But, in the case of immiscible fluids (i.e., $\sigma \neq 0$), the numerical value of D remains controversial. The aim of this paper is to present a mean-field theory. Treating the RVFP as a mass fractal, we predict that asymptotically, i.e., in the limit $A \to \infty$, the effective fractal dimension approaches the limiting value $D=2$; this prediction agrees with the results of recent computer simulation [7]. Moreover, the relation between σ and the onset of the asymptotic behavior as well as the trend of variation of the effective fractal dimension with time in the transient regime, for fixed σ, revealed by our analysis, are shown to be consistent with the corresponding results of computer simulation [7]. We also discuss the reasons for a smaller value of D observed in some laboratory experiments [8]. So far as the effects of the interfacial tension on the fractal dimension of RVFP are concerned, three possible scenarios emerge by examining the previous works: (i) $D = d_{\mathrm{DLA}}$ for all finite σ, including $\sigma = 0$; (ii) Maher and co-workers [8] claimed that their experimental data indicate $D = 1.79 \pm 0.04 \neq d_{\mathrm{DLA}}$ for $\sigma \neq 0$; (iii) Jasnow and Yeung [7] extrapolated the effective fractal dimension obtained from their computer simulation to the limit of large area and suggested that in the asymptotic regime $D = 2$ for $\sigma \neq 0$, i.e., asymptotically the RVFP becomes compact if $\sigma \neq 0$. Our aim is to find out, through mean-field arguments, which of the above-mentioned scenarios is actually followed by Nature.

To our knowledge, the first attempt to develop a mean-field theory for the RVFP was made by Sarkar [9]. His calculation was based on a simplified geometrical construction to describe the formation of fingers and splitting ("bifurcation") of the fingers. Through a particular choice of the numerical value of an adjustable parameter in his theory, Sarkar [9] suggested that D should lie between 1.77 and 1.86. Here we propose a different mean-field approach that does not require any such specific geometrical construction for describing finger formation and tip splitting. Our result for D is different from that suggested by Sarkar.

For completely miscible incompressible fluids, the dynamics of the RVFP is governed by the Laplace equation for pressure P, i.e.,

$$\nabla^2 P = 0 , \tag{1a}$$

together with the boundary condition

$$P = 0 . \tag{1b}$$

Similarly, the equations governing the growth of the DLA in the steady state is given by

$$\nabla^2 u = 0 , \tag{2a}$$

together with the boundary condition

$$u = 0 , \tag{2b}$$

where $u(r,t)$ is the probability that a random walker is at the point r at time t. Thus, formally, the two problems are equivalent [10]. Therefore, it is not surprising that experimentally one finds $D = d_{\mathrm{DLA}}$ when $\sigma = 0$. In fact, the random-walk algorithm for DLA, which is equivalent to solving Eqs. (2), has been used [5] to solve the problem

of viscous fingering in a Hele-Shaw cell described by Eqs. (1). Furthermore, the random-walk algorithm for DLA has been modified to incorporate the effects of interfacial tension in the numerical studies of viscous fingering [11]. In this work we take a similar approach; we extend Hentschel's [12] mean-field theory for DLA to develop a mean-field theory for RVFP by incorporating the effects of the nonvanishing interfacial tension in a nontrivial manner.

Hentschel's mean-field theory [12] for the fractal dimension of DLA is based on a Flory-type analysis, where some key ingredients are drawn from the well-known Flory arguments for branched polymers [13]. Recall that in the Flory theory for polymers the free energy consists of two competing terms, namely, a repulsive term $F_{\rm rep}$ arising from the monomer-monomer interaction and the "elastic energy term," which has an entropic origin. Our strategy is to identify the analog of a monomer in the RFVP and then to write down the expression for the effective free energy by drawing an analogy with the branched polymers. It is worth mentioning here that the effective free energy we write for the RVFP is only an analog of the free energy for polymers, as we are now considering a nonequilibrium growth process. However, the physical meaning of the effective free energy for the RVFP is that, if we grow this pattern a large number of times, the most probable configuration corresponds to the minimum of the effective free energy just as the most probable configuration of the polymers corresponds to the minimum of the corresponding free energy.

The area covered by the less viscous fluid in the RVFP corresponds to the diffusion-limited aggregate in our approach. A constant flux condition for the viscous finger experiments in circular geometry corresponds to a constant average number of random walkers sent in per unit time. Following Hentschel [12], we introduce a screening length l; this is the length up to which a random walker can penetrate inside the RVFP from the outside edges. Hentschel estimated this length scale in a mean-field approximation. Let all the N particles of the RVFP be smeared out uniformly over a volume R^d where d is the dimensionality of space. Then the probability that a lattice site is occupied inside the RVFP is $p=N/R^d$. Suppose the random walker makes N_w steps inside the pattern before it hits the interface. From the property of random walks, $N_w \sim l^2$. Since the random walker hits the interface and gets captures, on the average, after taking N_w steps, $N_w p \sim 1$. Therefore, the screening length is given by

$$l \sim N_w^{1/2} \sim (R^2/N)^{1/2} \sim R^{(2-D)/2} . \tag{3}$$

Now we assume that the RVFP behaves like an "aggregate" of n blobs, each of linear size l, which are connected like a branched polymer. The assumption of equal linear size of the blobs should be treated as a mean-field approximation. Therefore,

$$nl^2 \sim N . \tag{4}$$

We now identify the blobs as the analogs of the monomers. Therefore, for a RVFP the analog of the elastic energy (entropy) is given by [12]

$$F_{\rm el} \sim R^2/n^{1/2}l^2 . \tag{5}$$

Similarly, the repulsive term in the effective free energy is [12]

$$F_{\rm rep} \sim n^2/R^2 . \tag{6}$$

Next we introduce a term to account for the effects of the interfacial tension. This term is given by the product of the interfacial tension and the total area of the interface between the two fluids. However, because of the uniform width of the Hele-Shaw cell, the total interfacial area is directly proportional to the total perimeter of the two-dimensional RVFP. The total length of the perimeter L measured in units of a length scale a is given by $L=a(R/a)^D$. If we assume that the width of a typical finger is the smallest length scale in the problem, then measuring the length l in the units of the finger width gives us

$$L \sim R^D . \tag{7}$$

Therefore, the contribution from the interfacial tension to the effective free energy is given by

$$F_{\rm it} \sim TR^D , \tag{8}$$

where T is the dimensionless interfacial tension. Combining (5), (6), and (8), the total effective free energy is given by

$$F \sim [R^2/\{n^{1/2}l^2\}+n^2/R^2+TR^D] . \tag{9}$$

We can reexpress Eq. (9) in terms of only R and N by substituting for n and l from Eqs. (3) and (4), getting

$$F \sim [R+R^{-6}N^4+TR^D] . \tag{10}$$

Minimization of the effective free energy (10) with respect to R leads to

$$1+DTR^{(D-1)}=6R^{-7}N^4 . \tag{11}$$

Next, taking the natural logarithm of both sides of (11) and then differentiating $\ln N$ with respect to $\ln R$ and, finally, using $N \sim R^D$, we get

$$7-4D+(D-1)[1+\{TDR^{(D-1)}\}^{-1}]^{-1}=0 . \tag{12}$$

Equation (12) determines D for given T and R. Of course, if D varies with R, it is more appropriate to call it an effective fractal dimension, denoted by $D_{\rm eff}$; from now on, the symbol D will be reserved for the true asymptotic fractal dimension to distinguish it from $D_{\rm eff}$.

Before analyzing the full Eq. (12), consider its implications in two limiting situations: (a) for completely miscible fluids (i.e., $T=0$), Eq. (12) reduces to $7-4D=0$, i.e., $D=1.75$, the well-known [12] mean-field estimate of $d_{\rm DLA}$; (b) in the limit of large interfacial tension ($T \to \infty$), Eq. (12) leads to $D=2$, independent of R. Thus, according to our theory, the RVFP is compact during all stages of its evolution, provided $T \to \infty$. This is consistent with intuitive expectation as interfacial tension tends to fill out holes in the pattern.

For all finite nonzero values of T, the effective fractal dimension D_{eff}, obtained by numerically solving Eq. (12) depends on the choice of R. Indeed, for a given T, D_{eff} is a monotonically increasing function of R (see Fig. 1). It is interesting to note that D_{eff} approaches the limiting value $D=2$ in the asymptotic regime for all nonzero T. Although this prediction is based on mean-field arguments, this is consistent with the results of computer simulation [7] (see Fig. 6 of Ref. [7]). Moreover, for an evolving RVFP, the smaller T is the longer the transient regime is. These are the main results of this paper.

Next we present a quantitative treatment of the crossover from DLA to $D=2$. Introducing a variable $X=TDR^{D-1}$, we can reexpress Eq. (12) as

$$X=(7-4D)/(3D-6) . \tag{13}$$

Identifying $D=1.875$ (i.e., midway between $D=1.75$ and $D=2$) as the approximate location of the crossover point, we find from Eq. (13) that the location of the crossover point is given by

$$X=\tfrac{4}{3} . \tag{14}$$

Thus, for a given value of T, the location of the crossover point in Fig. 1 is given by the relation

$$\log_{10}R=(\tfrac{1}{0.875})[\log_{10}(\tfrac{32}{45})-\log_{10}T] . \tag{15}$$

The crossover from DLA to $D=2$, described by Eq. (15), is another quantitative prediction of our theory. For example, for the three different values of T, namely, $T_1=10^{-1}$, $T_2=10^{-5}$, and $T_3=10^{-8}$, the corresponding approximate values of the crossover radii given by Eq. (15) are, respectively, $9.4\lambda(T_1)$, $3.5\times10^{4}\lambda(T_2)$, and $9.4\times10^{8}\lambda(T_3)$, where the finger width $\lambda(T)$ is an increasing function of T.

So far as the measurement of D is concerned, for not-so-large values of T, the asymptotic regime may be well beyond the normal length and time scales of the laboratory experiments [8] on RFVP. This, we believe, is the reason for the difficulty in extracting the asymptotic fractal dimension of RVFP from laboratory experiments and computer simulation.

Maher and co-workers [8] did not vary the interfacial tension directly in their experiments. Instead, by varying the flow rate Q, they varied the dimensionless driving force $C=Q\mu/(b\sigma)$, where μ is the viscosity and b is the gap between the two plates. A larger flow rate is effectively equivalent to a smaller σ and vice versa. At first sight, it may be tempting to compare our prediction for large T with the experimental data for small C. Although the experimentally observed value for D_{eff} in the latter regime is, indeed, 2, it must not be regarded as evidence in favor of our prediction. Small C corresponds to large finger width λ and, hence, the compactness of the RVFP observed by Maher and co-workers in this regime is a consequence of the fact that the radial size of the pattern is much smaller than λ. On the other hand, our theory has been developed for the opposite limit, namely, radial size of the pattern much larger than λ. Therefore, our theoretical prediction cannot be compared directly with the experimental data of Maher and co-workers [8]. We urge our experimentalist colleagues to test our prediction either by directly varying the interfacial tension or by using two strongly immiscible fluids (i.e., large σ), preferably in larger Hele-Shaw cells so as to ensure the probing of the true asymptotic regime.

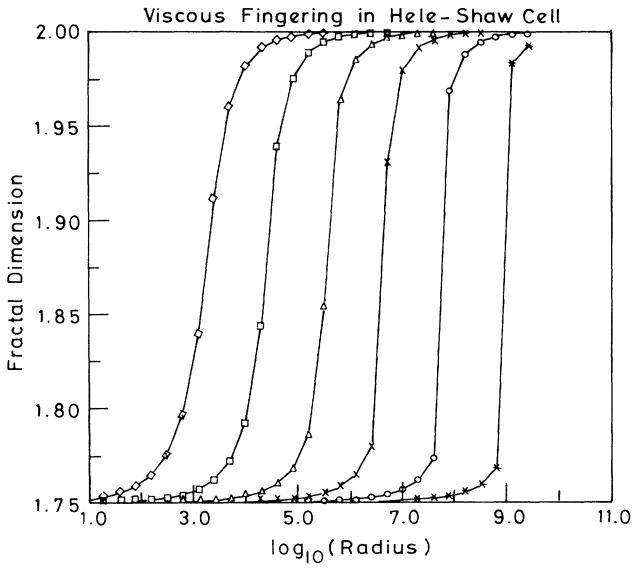

FIG. 1. The effective fractal dimension D_{eff}, for the viscous fingering in a Hele-Shaw cell, plotted as a function of the logarithm of linear size ("radius") R of the RVFP for four different values of the dimensionless interfacial tension T. The symbols *, ○, ×, △, □, and ◇ correspond to $T=10^{-8}$, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, and 10^{-3}, respectively.

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